DEPENDENCE ON THE SPIN STRUCTURE OF THE ETA AND ROKHLIN INVARIANTS

MATTIAS DAHL

ABSTRACT. We study the dependence of the eta invariant η_D on the spin structure, where D is a twisted Dirac operator on a $(4k+3)$-dimensional spin manifold. The difference between the eta invariants for two spin structures related by a cohomolgy class which is the reduction of a $H^1(M, \mathbb{Z})$-class is shown to be a half integer. As an application of the technique of proof the generalized Rokhlin invariant is shown to be equal modulo 8 for two spin structures related in this way.

1. INTRODUCTION

1.1. **The eta invariant.** Let M be a compact manifold of dimension $4k + 3$. Let D be a self-adjoint first order elliptic operator on M. The eta invariant of D is defined as

$$\eta_D = \frac{1}{2}\Big(\eta(0) + \dim \ker D\Big)$$

where the eta-function $\eta(s)$ is given by

$$\eta(s) = \sum_{\lambda \in \operatorname{spec} D, \lambda \neq 0} \frac{\operatorname{sign} \lambda}{|\lambda|^s}$$

which converges for $\operatorname{Re} s$ large, and has an analytic continuation which is regular at $s = 0$, see [5].

Assume that M is a spin manifold and let σ be a spin structure on M with associated spinor bundle S_σ. Let E be a vector bundle with connection on M. We use the notation $\eta(\sigma; E)$ for the eta invariant of the twisted Dirac operator D^E on $S_\sigma \otimes E$. The set $\operatorname{Spin}(M)$ of spin structures on M is an affine space modeled on $H^1(M; \mathbb{Z}_2)$, i.e. the vector space $H^1(M; \mathbb{Z}_2)$ acts freely and transitively on $\operatorname{Spin}(M)$:

$$\operatorname{Spin}(M) \times H^1(M; \mathbb{Z}_2) \ni (\sigma, \delta) \mapsto \sigma + \delta \in \operatorname{Spin}(M).$$

For $\sigma \in \operatorname{Spin}(M)$ and $\delta, \delta' \in H^1(M; \mathbb{Z}_2)$ we define the first and second difference functions of $\eta(\sigma; E)$ by

$$\Delta\eta(\sigma, \delta; E) = \eta(\sigma + \delta; E) - \eta(\sigma; E)$$

and
$$\Delta^2\eta(\sigma, \delta', \delta; E) = \Delta\eta(\sigma + \delta', \delta; E) - \Delta\eta(\sigma, \delta; E).$$
The first difference $\Delta\eta$ is a special case of the relative eta invariant introduced by Atiyah, Patodi and Singer in [2] and since studied by many authors. The ideas behind the following theorem also goes back to Atiyah, Patodi and Singer.

Theorem 1.1. *Let (M, g) be a compact spin manifold of dimension $4k + 3$. Let σ be a spin structure on M and let $\delta, \delta' \in H^1(M; \mathbb{Z}_2)$. Suppose δ is the reduction modulo 2 of an integer class. Let E be a vector bundle on M with connection ∇^E. Then*

(a) $\Delta\eta(\sigma, \delta; E) \in \frac{1}{2}\mathbb{Z}$,
(b) $\Delta^2\eta(\sigma, \delta', \delta; E) \in \mathbb{Z}$.

These differences do not depend on the metric g.

1.2. **The Rokhlin invariant.** Let M be a compact spin manifold of dimension $8k + 3, k \geq 0$, and suppose that M is a boundary. The Rokhlin invariant is a function defined on the set $\mathrm{Spin}(M)$ of spin structures on M and taking values in the integers modulo 16,
$$R_M : \mathrm{Spin}(M) \to \mathbb{Z}_{16},$$
see [13] and [3]. Given a spin structure σ on M, we take a spin manifold N with spin structure τ such that (M, σ) is the boundary of (N, τ). Such an N exists since the forgetful homomorphism of cobordism rings $\Omega_{8k+3}^{\mathrm{Spin}} \to \Omega_{8k+3}^{SO}$ is injective, see [11, p.351], and thus M is an oriented boundary if and only if it is a spin boundary. The Rokhlin function is defined by
$$R_M(\sigma) = \mathrm{Sign}(N) \mod 16$$
where $\mathrm{Sign}(N)$ is the signature of N. This is independent of the choice of N since the signature invariant is additive and by a theorem of Ochanine, see [6, p.113], the signature of a closed $8k + 4$ dimensional spin manifold is a multiple of 16.

Lee and Miller [10] showed that if we equip M with a riemannian metric then the Rokhlin function can be computed in terms of eta invariants of the signature operator and of twisted Dirac operators,

(1) $$R_M(\sigma) = -\eta_{\mathrm{Hirz}} + 8\sum_i b_i \eta_{D^{Z_i}} \mod 16,$$

where b_i are integers and Z_i are vector bundles. In dimension 3 this formula is

(2) $$R_M(\sigma) = -\eta_{\mathrm{Hirz}} - 8\eta_D \mod 16,$$

and in dimension 11 we have

(3) $$R_M(\sigma) = -\eta_{\mathrm{Hirz}} + 8\eta_{D^{TM}} - 32\eta_D \mod 16.$$

Since the expression on the right-hand side of (1) is defined even if M is not a boundary, we take (1) as the definition of R_M for a general $8k + 3$-dimensional

spin manifold. From (1) it also follows that this is independent of the riemannian metric, but a priori this extension of the Rokhlin function takes it values in $\mathbb{R}$ modulo 16. A theorem by Fischer and Kreck [4] tells us that (1) is always an integer modulo 16.

In this paper we are going to use (1) to study how the Rokhlin invariant varies with the spin structure on M. For $\sigma \in \mathrm{Spin}(M)$ and $\delta_0, \ldots, \delta_m \in H^1(M; \mathbb{Z}_2)$ the difference functions of R_M are defined inductively by

$$\Delta^1 R_M(\sigma, \delta_0) = R_M(\sigma + \delta_0) - R_M(\sigma),$$
$$\Delta^{m+1} R_M(\sigma, \delta_0, \ldots, \delta_m) = \Delta^m R_M(\sigma + \delta_0, \delta_1, \ldots, \delta_m)$$
$$- \Delta^m R_M(\sigma, \delta_1, \ldots, \delta_m).$$

In [13] Turaev shows that $\Delta^4 R_M = 0$ for any 3-dimensional manifold M. Finashin conjectures in [3] that $\Delta^4 R_M = 0$ holds in all dimensions.

Our main result is the following.

Theorem 1.2. *If $\delta \in H^1(M; \mathbb{Z}_2)$ is the reduction modulo 2 of an integer class then $\Delta R_M(\sigma, \delta) = 0$ or $8 \mod 16$.*

This generalizes a result by Taylor, see [12, Theorem 6.2.], in dimension 3. In the case where M is a boundary this follows by results of Finashin in [3] but in the general case it seems to be new.

2. Spin structures and Dirac operators

Throughout this paper (M, g) will denote a compact oriented riemannian spin manifold of dimension n. For the necessary background on spin geometry we refer to [5] and [7].

2.1. **Spin structures and spinor bundles.** Let $\pi : \mathrm{Spin}(n) \to SO(n)$ be the double cover of the group $SO(n)$. A spin structure σ on M is a $\mathrm{Spin}(n)$-principal bundle over M which is pointwise a double cover of the oriented orthonormal frame bundle $SO(M)$,

$$\xi : \sigma \to SO(M),$$

$$\xi(pg) = \xi(p)\pi(g), \quad p \in \sigma, g \in \mathrm{Spin}(n).$$

We denote the set of (isomorphism classes of) spin structures by $\mathrm{Spin}(M)$. This is an affine space modeled on $H^1(M; \mathbb{Z}_2)$, given two spin structures σ, σ' there is an element $\sigma' - \sigma = \delta \in H^1(M; \mathbb{Z}_2)$ which is the difference of the spin structures. This difference element can be constructed as follows. Let $\{U_i, a_{ij}\}$ be a trivialization of the bundle $SO(M)$ with transition functions

$$a_{ij} : U_i \cap U_j \to SO(n)$$

and let

$$b_{ij}, b'_{ij} : U_i \cap U_j \to \mathrm{Spin}(n)$$

be the transition functions for σ, σ'. Then

$$\pi \circ b_{ij} = \pi \circ b'_{ij} = a_{ij}$$

and the c_{ij} defined as

(4) $$c_{ij} = b'_{ij} b_{ij}^{-1} : U_i \cap U_j \to \ker(\pi) = \{\pm 1\} = \mathbb{Z}_2$$

give $\delta \in H^1(M; \mathbb{Z}_2)$. The element δ can be viewed as a $\mathbb{Z}_2$-principal fibre bundle. By the action of $\mathbb{Z}_2 = \{\pm 1\}$ on $\mathbb{C}$ we get an associated flat complex line bundle which we denote by L_δ.

Let $\rho : \mathrm{Spin}(n) \to \mathrm{End}(\mathcal{S})$ be the spinor representation of the spin group on the vector space $\mathcal{S}$. We denote by S_σ the spinor bundle $\sigma \times_\rho \mathcal{S}$ associated to σ. In the trivialization introduced above S_σ will have transition functions $\rho(b_{ij}) : U_i \cap U_j \to \mathrm{End}(\mathcal{S})$ and from (4) it follows that the transition functions for $S_{\sigma'}$ are

$$\rho(b'_{ij}) = \rho(c_{ij})\rho(b_{ij}).$$

Since $\rho(-1) = -1$ this translates to

(5) $$S_{\sigma'} = L_\delta \otimes S_\sigma.$$

In this paper we will be interested in the case when the line bundle L_δ is topologically trivial, that is when L_δ has a non-vanishing section. A smooth non-vanishing section l of L_δ defines an invertible map

$$l : \varphi \mapsto l \otimes \varphi$$

which sends sections of S_σ to sections of $S_{\sigma'}$. If we normalize l to $|l| = 1$ this map is pointwise an isometry since

$$\langle l \otimes \varphi, l \otimes \psi \rangle = |l|^2 \langle \varphi, \psi \rangle$$

and also a surjective isometry of the Hilbert spaces of square integrable sections

$$l : L^2(S_\sigma) \to L^2(S_{\sigma'}).$$

Let $H^1(M; \mathbb{Z}_2)_0$ be the subspace of $H^1(M; \mathbb{Z}_2)$ consisting of δ such that L_δ is topologically trivial. It can be characterized as follows, see [7, p.84] and [1].

Proposition 2.1. $\delta \in H^1(M; \mathbb{Z}_2)_0$ *if and only if there is an integral* $\alpha \in H^1(M; \mathbb{R})$ *such that*

$$\delta([\gamma]) = e^{\pi i \int_\gamma \alpha}$$

for all closed curves γ.

Proof. Suppose l is a section of L_δ with $|l| = 1$. Such a section is the same as a δ-equivariant function $\tilde{l} = e^{\pi i \theta}$ on the universal cover $\widetilde{M}$ of M. Let $\alpha = d\theta =$

$\frac{1}{\pi i} l^{-1} dl$. Then α drops down to a closed real-valued one-form on M and

$$
\begin{aligned}
e^{\pi i \int_\gamma \alpha} &= e^{\pi i \int_{\tilde\gamma} d\theta} \\
&= e^{\pi i \left(\theta(\gamma(x_0)) - \theta(x_0) \right)} \\
&= \tilde{l}(\gamma(x_0)) \tilde{l}(x_0)^{-1} \\
&= \delta([\gamma])
\end{aligned}
$$

where $\tilde\gamma$ is the lift of γ to $\widetilde{M}$, a path from x_0 to $\gamma(x_0)$. On the other hand, suppose $\delta([\gamma]) = e^{\pi i \int_\gamma \alpha}$ for some closed one-form α. Then the pullback of α to $\widetilde{M}$ is the differential of some function θ and $\tilde{l} = e^{\pi i \theta}$ is a δ-equivariant function on $\widetilde{M}$, which gives a non-vanishing section of L_δ. $\qquad \square$

2.2. **Dirac operators.** For a spin structure σ with associated spinor bundle S_σ the Dirac operator D is defined by

$$
D\varphi = e_i \cdot \nabla_{e_i} \varphi
$$

where e_i is a local orthonormal frame, the dot is Clifford multiplication, and φ is a smooth section of S_σ. D extends to a self-adjoint operator on $L^2(S_\sigma)$. As before we let σ and $\sigma' = \sigma + \delta$ be spin structures on M and we assume that $\delta \in H^1(M; \mathbb{Z}_2)_0$. We let l be a unit norm section of L_δ and define a connection ∇' by

$$
(6) \qquad\qquad \nabla'\varphi = l^{-1}\nabla(l\varphi) = \nabla\varphi + l^{-1}dl\varphi
$$

acting on sections of S_σ. This connection is metric:

$$
\begin{aligned}
X\langle \varphi, \psi \rangle &= X\langle l\varphi, l\psi \rangle \\
&= \langle \nabla_X(l\varphi), l\psi \rangle + \langle l\varphi, \nabla_X(l\psi) \rangle \\
&= \langle l^{-1}\nabla_X(l\varphi), \psi \rangle + \langle \varphi, l^{-1}\nabla_X(l\psi) \rangle \\
&= \langle \nabla'_X \varphi, \psi \rangle + \langle \varphi, \nabla'_X \psi \rangle.
\end{aligned}
$$

Let D' be the operator of Dirac type defined using ∇',

$$
D' = e_i \cdot \nabla'_{e_i} = l^{-1} D l = D + l^{-1}\operatorname{grad} l \cdot .
$$

The following is a fundamental observation.

Proposition 2.2. D' acting on $L^2(S_\sigma)$ is isospectral to D acting on $L^2(S_{\sigma'})$.

Proof. Suppose φ is a section of $S_{\sigma'}$ and an eigenspinor of D with eigenvalue λ. Then

$$
D'(l^{-1}\varphi) = l^{-1}D\varphi = \lambda l^{-1}\varphi
$$

so $l^{-1}\varphi$ is an eigenspinor of D', also with eigenvalue λ. If ψ is an eigenspinor of D' then $l\psi$ is an eigenspinor of D with the same eigenvalue. $\qquad \square$

This means that D' acting on $L^2(S_\sigma)$ has the same spectral invariants as D acting on $L^2(S_{\sigma'})$.

3. The eta invariant

In this section we prove the theorem on the eta invariant stated in the introduction and state some corollaries.

Theorem 3.1. *Let (M, g) be a compact spin manifold of dimension $4k + 3$. Let σ be a spin structure on M and let $\delta \in H^1(M; \mathbb{Z}_2)_0$, $\delta' \in H^1(M; \mathbb{Z}_2)$. Let E be a vector bundle on M with connection ∇^E. Then*

  (a) $\Delta\eta(\sigma, \delta; E) \in \frac{1}{2}\mathbb{Z}$,
  (b) $\Delta^2\eta(\sigma, \delta', \delta; E) \in \mathbb{Z}$.

  These differences do not depend on the metric g.

Proof. As before let l be a normalized section of L_δ and let $\nabla' = l^{-1}\nabla l$. Let $\overline{M} = M \times I$ where $I = [0, 1]$ and denote by ν the vectorfield $\frac{\partial}{\partial t}$ on $\overline{M}$. The spin structure σ on M induces a spin structure on $\overline{M}$, the associated spinor bundle is equal to $S_\sigma \oplus S_\sigma$. To shorten the notation we will write this as $\overline{S}$.

Let $\chi : I \to I$ be a smooth function such that $\chi(t) = 0, t \leq 1/3$ and $\chi(t) = 1, t \geq 2/3$. Define a connection $\widetilde{\nabla}$ on sections of the trivial complex line bundle $\mathbf{1}$ over $\overline{M}$ by

$$\widetilde{\nabla}_X f = \left(X + \chi(t)l^{-1}dl(X)\right)f, \quad \widetilde{\nabla}_\nu f = \nu f$$

where $X \in TM$. This connection is metric and the induced connection $\nabla \otimes \widetilde{\nabla}$ on $\overline{S} \otimes \mathbf{1}$ interpolates between ∇ close to $t = 0$ and ∇' close to $t = 1$.

Let $\overline{D}^E$ be the Dirac operator on $\overline{S} \otimes E \otimes \mathbf{1}$ constructed using the connection $\nabla \otimes \nabla^E \otimes \widetilde{\nabla}$. Close to the boundary component $M \times \{0\}$ of $\overline{M}$ we have $\overline{D}^E = \nu \cdot (\frac{\partial}{\partial t} + D^E)$ and close to $M \times \{1\}$ we have $\overline{D}^E = \nu \cdot (\frac{\partial}{\partial t} + D'^E)$.

The Atiyah-Patodi-Singer index theorem [5, p.305] tells us that

$$\text{(7)} \qquad \text{ind}(\overline{D}^E) = \int_{\overline{M}} \widehat{A}(M) \wedge \text{ch}(\nabla^E) \wedge \text{ch}(\widetilde{\nabla}) - \eta_{D^E} + \eta_{D'^E}.$$

where the index on the left hand side is the Fredholm index of $\overline{D}^E$ acting on sections of $\overline{S} \otimes \mathbf{1} \otimes E$ satisfying the Atiyah-Patodi-Singer boundary condition.

Since D'^E acting on sections of $S_\sigma \otimes E$ is isospectral to D^E acting on sections of $S_{\sigma+\delta} \otimes E$ we have $\eta_{D'^E} - \eta_{D^E} = \eta(\sigma + \delta; E) - \eta(\sigma; E) = \Delta\eta(\sigma, \delta; E)$ and

$$\text{(8)} \qquad \Delta\eta(\sigma, \delta; E) = -\text{ind}(\overline{D}^E) + \int_{\overline{M}} \widehat{A}(M) \wedge \text{ch}(\nabla^E) \wedge \text{ch}(\widetilde{\nabla}).$$

Next we compute $\text{ch}(\widetilde{\nabla})$ and for this we need the curvature $\widetilde{R}$ of $\widetilde{\nabla}$. For $X \in TM$ we write

$$\widetilde{\nabla}_X = X + \chi(t)l^{-1}dl(X) = \nabla + \pi i\chi(t)\alpha(X)$$

where as in the proof of Proposition (2.1) $\alpha = \frac{1}{\pi i} l^{-1} dl$ is a closed integer-valued one-form. For $X, Y \in TM$ we have

$$
\begin{aligned}
\widetilde{\nabla}_X \widetilde{\nabla}_Y &= \left(X + \pi i \chi(t) \alpha(X)\right)\left(Y + \pi i \chi(t) \alpha(Y)\right) \\
&= XY + \pi i \chi(t)\left(X(\alpha(Y)) + \alpha(Y)X + \alpha(X)Y\right) \\
&\quad + \left(\pi i \chi(t)\right)^2 \alpha(X)\alpha(Y)
\end{aligned}
$$

so

$$
\begin{aligned}
\widetilde{R}(X, Y) &= \widetilde{\nabla}_X \widetilde{\nabla}_Y - \widetilde{\nabla}_Y \widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]} \\
&= \pi i \chi(t)\left(X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])\right) \\
&= \pi i \chi(t) d\alpha(X, Y) \\
&= 0
\end{aligned}
$$

since α is closed. Next we have

$$
\begin{aligned}
\widetilde{R}(\nu, X)\varphi &= (\widetilde{\nabla}_\nu \widetilde{\nabla}_X - \widetilde{\nabla}_X \widetilde{\nabla}_\nu)\varphi \\
&= \pi i \chi'(t)\alpha(X)
\end{aligned}
$$

and we conclude that

$$
\widetilde{R} = \pi i d\chi \wedge \alpha.
$$

Thus we find that the Chern character of $\widetilde{\nabla}$ is

$$
(9) \qquad \mathrm{ch}(\widetilde{\nabla}) = \mathrm{tr}\, e^{\frac{i\widetilde{R}}{2\pi}} = e^{-\frac{d\chi \wedge \alpha}{2}} = 1 - \frac{d\chi \wedge \alpha}{2}
$$

and we see that the integral in (7) is

$$
\int_{\overline{M}} \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge \mathrm{ch}(\widetilde{\nabla}) = \int_{\overline{M}} \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge (1 - \frac{d\chi \wedge \alpha}{2}).
$$

When we integrate over the I factor only the term with $d\chi$ will contribute and we have left

$$
(10) \qquad \int_{\overline{M}} \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge \mathrm{ch}(\widetilde{\nabla}) = -\frac{1}{2} \int_M \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge \alpha.
$$

The same calculation can now be used again to see that this integral is the index of a twisted Dirac operator. Let S^1 be the circle of length 1. Then

$$
\begin{aligned}
\int_M \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge \alpha &= \int_{M \times S^1} \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge dt \wedge \alpha \\
&= \int_{M \times S^1} \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge (1 + dt \wedge \alpha) \\
&= \int_{M \times S^1} \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge e^{dt \wedge \alpha}.
\end{aligned}
$$

Since $dt \wedge \alpha \in H^2(M \times S^1; \mathbb{Z})$ there is a complex line bundle $K \to M \times S^1$ with $c_1(K) = dt \wedge \alpha$, curvature 2-form $\frac{2\pi}{i} dt \wedge \alpha$, and $\mathrm{ch}(\nabla^K) = e^{dt \wedge \alpha}$. This means that (10) is equal to

$$
\begin{aligned}
-\frac{1}{2} \int_M \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge \alpha &= -\frac{1}{2} \int_{M \times S^1} \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge \mathrm{ch}(\nabla^K) \\
&= -\frac{1}{2} \mathrm{ind}(D^{E \otimes K})
\end{aligned}
$$

(11)

From (8) and (11) we now get

(12) $$\Delta \eta(\sigma, \delta; E) = -\mathrm{ind}(\overline{D}^E) - \frac{1}{2} \mathrm{ind}(D^{E \otimes K}),$$

which is a half integer. We have proved part (a) of the theorem. To prove part (b) note that the last term,

$$\mathrm{ind}(D^{E \otimes K}) = \int_M \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge \alpha,$$

in (12) does not depend on the spin structure σ. When we take a second difference,

$$\Delta^2 \eta(\sigma, \delta', \delta; E) = \Delta \eta(\sigma + \delta', \delta; E) - \Delta \eta(\sigma, \delta; E),$$

these terms will cancel, and part (b) follows. $\qquad \square$

In the case where the twisting bundle E is flat, for example if E is the trivial line bundle, we get the following corollary.

Corollary 3.2. *Let (M, g) be a compact spin manifold of dimension $4k + 3$. Let σ be a spin structure on M and suppose $\delta \in H^1(M; \mathbb{Z}_2)_0$. Let E be a vector bundle on M with a flat connection ∇^E. Then $\Delta \eta(\sigma, \delta; E) \in \mathbb{Z}$,*

Proof. Since $\mathrm{ch}(\nabla^E) = 1$ we have from (8) and (10)

$$
\begin{aligned}
\Delta \eta(\sigma, \delta; E) &= -\mathrm{ind}(\overline{D}) - \frac{1}{2} \int_M \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge \alpha \\
&= -\mathrm{ind}(\overline{D}) - \frac{1}{2} \int_M \widehat{A}(M) \wedge \alpha.
\end{aligned}
$$

The integrand $\widehat{A}(M) \wedge \alpha$ only contains terms of degrees $4p+1$, $p \geq 0$ so the integral over M which has dimension $4k + 3$ vanishes, and the corollary follows. $\qquad \square$

For a self-adjoint first order elliptic operator D the reduced eta invariant is defined by

$$\overline{\eta}_D = \eta_D \mod 1 \in \mathbb{R}/\mathbb{Z}.$$

We use the same notation as above for the reduced eta invariant of twisted Dirac operators, and the for their difference functions with respect to variations of the spin structure. In terms of $\overline{\eta}$ we can formulate the following corollary.

Corollary 3.3. *Under the conditions in Theorem 3.1 we have*

(a) $\Delta\overline{\eta}(\sigma, \delta; E) = -\frac{1}{2}\int_M \widehat{A}(M) \wedge \mathrm{ch}(\nabla^E) \wedge \alpha \in \frac{1}{2}\mathbb{Z}$,
(b) $\Delta^2\overline{\eta}(\sigma, \delta', \delta; E) = 0$.

4. THE ROKHLIN FUNCTION

We are now going to prove the theorem on the Rokhlin invariant stated in the introduction. We begin by stating in more detail the formula by Lee and Miller expressing the Rokhlin function in terms of eta invariants.

Theorem 4.1. [10], [8], [9]

$$(13) \qquad R_M(\sigma) = -\eta_{\mathrm{Hirz}} + 8\sum_i b_i\eta(\sigma; Z_i) \quad \mathrm{mod}\ 16,$$

where the b_i are integers and the Z_i are tensor bundles, that is the Z_i are bundles associated to the orthonormal frame bundle $SO(M)$ through representations ρ_i of $SO(n)$.

This theorem is based on the following formula expressing the L-genus in terms of twisted $\widehat{A}$-genera.

$$(14) \qquad L(M) = 8\sum_i b_i\widehat{A}(M)\,\mathrm{ch}(\nabla^{Z_i}).$$

Using Proposition 2.1 we see that Theorem 1.2 in the introduction is equivalent to the following.

Theorem 4.2. *If $\delta \in H^1(M; \mathbb{Z}_2)_0$ then $\Delta R_M(\sigma, \delta) = 0 \ \mathrm{mod}\ 8$.*

Proof. From (13) we have

$$\Delta R_M(\sigma, \delta) = 8\sum_i b_i\Delta\eta(\sigma, \delta; Z_i) \quad \mathrm{mod}\ 16.$$

Equations (8) and (10) tells us that

$$\Delta R_M(\sigma, \delta) = -8\sum_i b_i\,\mathrm{ind}(\overline{D}^{Z_i}) - \frac{1}{2}\sum_i 8b_i\int_M \widehat{A}(M) \wedge \mathrm{ch}(\nabla^{Z_i}) \wedge \alpha \quad \mathrm{mod}\ 16$$

and using (14) we see that

$$\Delta R_M(\sigma, \delta) = -8\sum_i b_i\,\mathrm{ind}(\overline{D}^{Z_i}) - \frac{1}{2}\int_M L(M) \wedge \alpha \quad \mathrm{mod}\ 16.$$

Since M has dimension $8k+3$ and $L(M)\wedge\alpha$ only contains terms of degrees $4p+1$, $p \geq 0$, the integral vanishes and we are left with

$$\Delta R_M(\sigma, \delta) = -8\sum_i b_i\,\mathrm{ind}(\overline{D}^{Z_i}) \quad \mathrm{mod}\ 16$$

which proves the theorem. $\square$

From the relations (2) and (3) we get the following corollary.

Corollary 4.3. *If $\delta \in H^1(M; \mathbb{Z}_2)_0$ and $\dim M = 3$ then*

$$\Delta R_M(\sigma, \delta) = -8\operatorname{ind}(\overline{D}) \mod 16$$

and if $\dim M = 11$ then

$$\Delta R_M(\sigma, \delta) = 8\operatorname{ind}(\overline{D}^{TM}) \mod 16$$

where $\overline{D}$ is the operator on $M \times I$ introduced in the proof of Theorem (3.1).

References

[1] B. Ammann, *Spin-Strukturen und das Spektrum des Dirac-Operators*, Ph.D. thesis, 1998.

[2] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. II*, Math. Proc. Cambridge Philos. Soc. **78** (1975), no. 3, 405–432.

[3] S. Finashin, *Variation of the Rokhlin function in higher dimensions*, Topology of manifolds and varieties, Amer. Math. Soc., Providence, RI, 1994, pp. 147–158.

[4] Thomas Fischer and Matthias Kreck, *Rohlin invariants, integrality of some spectral invariants and bordism of spin diffeomorphisms*, Topology Appl. **37** (1990), no. 2, 181–199.

[5] Peter B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, second ed., CRC Press, Boca Raton, FL, 1995.

[6] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung, *Manifolds and modular forms*, Friedr. Vieweg & Sohn, Braunschweig, 1992, With appendices by Nils-Peter Skoruppa and by Paul Baum.

[7] H. Blaine Lawson and Marie-Louise Michelsohn, *Spin geometry*, Princeton University Press, Princeton, NJ, 1989.

[8] Kefeng Liu, *Modular invariance and characteristic numbers*, Comm. Math. Phys. **174** (1995), no. 1, 29–42.

[9] Kefeng Liu and Wei Ping Zhang, *Elliptic genus and η-invariant*, Internat. Math. Res. Notices (1994), no. 8, 319–327.

[10] Edward Y. Miller and Ronnie Lee, *Some invariants of spin manifolds*, Topology Appl. **25** (1987), no. 3, 301–311.

[11] Robert E. Stong, *Notes on cobordism theory*, Princeton University Press, Princeton, N.J., 1968, Mathematical notes.

[12] Laurence R. Taylor, *Relative Rochlin invariants*, Topology Appl. **18** (1984), no. 2-3, 259–280.

[13] V. G. Turaev, *Cohomology rings, linking coefficient forms and invariants of spin structures in three-dimensional manifolds*, Mat. Sb. (N.S.) **120(162)** (1983), no. 1, 68–83, 143.

Department of Mathematics, Royal Institute of Technology, S-100 44 Stockholm, Sweden,

E-mail address: dahl@math.kth.se